JOURNAL OF APPROXIMATION THEORY 59, 12–37 (1989)

Upper Bound for the Degree of an Approximating Monomial

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Communicated by R. Bojanic

Received May 15, 1987

It is known that if P is any polynomial of degree $\leqslant n$ and $m(x)=cx^k$ is a monomial of best approximation to P in $L_p[a, b]$ among all monomials of degree $>n$, then

(i) if $p=\infty$, no upper bound for k exists, and

(ii) if $1\leqslant p<\infty$, there is $K_n=K_n(a, b; p)$ (independent of the polynomial P) such that

$$k\leqslant K_n. \tag{$*$}$$

The proof of the existence of the upper bound K_n is not constructive. In particular, with M_n denoting the best bound K_n (i.e., M_n is the infimum of all K_n for which $(*)$ is true), no estimate for M_n is available (for a general p). In this paper we have considered approximation by quasi-monomials cx^k (i.e., k is real and $\geqslant n$). We have obtained estimates for M_n for the case of the L_2-norm on the interval $[0, 1]$; our main result is

$$\tfrac{1}{4}(n+1)^3\leqslant M_n\leqslant 6(n+1)^3.$$

I. INTRODUCTION

The problem of approximation of polynomials by monomials was first investigated by B. M. Baishanski as a converse of G. G. Lorentz's problem of approximation of x^N by certain polynomials [1].

In [5] G. G. Lorentz conjectured the following: Among all polynomials of the form $p(x)=\sum_{i=1}^{s} a_i x^{k_i}$ $(0\leqslant l_1<k_2\cdots<k_s<N)$, where s is a fixed integer $<N$, the polynomial of best uniform approximation to x^N has powers $k_1=N-s$, $k_2=N-s+1, ..., k_s=N-1$.

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0021-9045/89 $3.00

This conjecture was proved by I. Borosh, C. K. Chui, and P. W. Smith [3]. They proved the following more general result:

THEOREM. *Let N, l, and k be fixed positive integers such that $l<N$ and $l\leqslant k$. Let $\lambda_1, ..., \lambda_k$ be integers such that*

$$0\leqslant\lambda_1<\cdots<\lambda_l<N<\lambda_{l+1}<\cdots<\lambda_k.$$

Among all polynomials $P(x)=\sum_{i=1}^{l} a_i x^{\lambda_i}$, the polynomial of best uniform approximation to x^N on $[0, 1]$ *has powers*

$$N-l, ..., N-1, \qquad N+1, ..., N+k-l.$$

In [10] P. W. Smith gave a proof (based on an observation of A. Pinkus) of the above result in any L_p-norm, $1\leqslant p\leqslant\infty$. See also [4, 9].

If the above problem of Lorentz is inverted, namely, if x^N is replaced by a polynomial P of degree $\leqslant n$ and P is approximated by monomials $m(x)=cx^k$, $k\geqslant 1$, then an analogue of the above result will not hold; for example if $P(x)=x^N-[N/(N+1)]\,x^{N-1}$, then among all monomials, the monomial of best L_2-approximation to P on $[0, 1]$ has power $=3N+1$ [1].

This led B. M. Baishanski [1] to the question of the existence of an upper bound for the best approximating monomials if P runs over the set of all polynomials of degree $\leqslant n$.

In [1] Baishanski stated the following general result and proved a special case of it, namely:

Let l be a fixed positive integer. If P is a polynomial of degree $\leqslant n$ and $Q(x)=\sum_{k=l}^{l} c_k x^{\lambda_k(P)}$ is a polynomial of length $\leqslant l$ (the length of a polynomial is the number of its non-zero coefficients) of best approximation to P in $L_p[a, b]$, $-\infty<a<b<\infty$, among all polynomials of length $\leqslant l$, then

(i) if $p=\infty$ and $2l\leqslant n+1$, no upper bound for $\lambda_l(P)$ (we assume $\lambda_1(P)<\cdots<\lambda_l(P)$) exists, and

(ii) if $1\leqslant p<\infty$, there is $K_n=K_n(a, b; l, p)$ such that

$$\lambda_l(P)\leqslant K_n.$$

A proof of (i) and the special case of (ii) when $p=2$, $[a, b]=[0, 1]$, and $l=1$ is given in [1], and a proof of (ii) is given in [2]. In fact, stronger results were obtained in [2]; for example,

THEOREM. *Let S be a set of non-negative integers, and denote by $\pi_{l-1}(S)$ $(l\geqslant 1)$ the collection of all polynomials of length $\leqslant l-1$ with exponents* *chosen from the set S. Let K be a compact set in* $L_p\,[a, b]$ $-\infty < a < b < \infty$, $1 \leqslant p < \infty$, *such that*

$$K \cap \pi_{l-1}(S) = \phi.$$

If

(i) $\sum_{s \in S} 1/(s+1) = \infty$ *and, in case* $p = 1$, *measure* $\{x : f(x) = g(x)\} = 0$ *for every* $f \in K$, $g \in \pi_{l-1}(S)$, *or if*

(ii) *every function in K is analytic on* $[a, b]$ *and, in case* $a = -b$, *S contains infinitely many odd and infinitely many even integers,*

then there exists $d = d(K, S, l)$ *such that, if* $f \in K$ *and P is a best approximation to f in* $\pi_l(S)$, *then* $\deg P \leqslant d$.

This is a pure existence theorem. The proof is not constructive and it gives no information about the value of d.

The question arises of obtaining an estimate of the degree of a best approximating polynomial of length $\leqslant l$, when a polynomial of degree $\leqslant n$ is being approximated. It is natural to restrict ourselves to a simple case, first, and we do this in this paper. Namely, we consider only the L_2-norm on $[0, 1]$, we consider only the length $l = 1$, and instead of approximating by monomials cx^k, k a non-negative integer, we approximate by quasi-monomials cx^t, t real and $\geqslant n$.

The results in this paper are from the author's doctoral dissertation written under the supervision of Professor Bogdan M. Baishanski at the Ohio State University.

II. Notation and the Main Theorem

π_n denotes the set of all real polynomials of degree $\leqslant n$ $(n \geqslant 0)$. If K is a set of real numbers, then $\|\cdot\|_K$ denotes the uniform norm on $K \cdot \|\cdot\|_2$ denotes the L_2-norm on $[0, 1]$.

For $P \in \pi_n$ and $t > -\frac{1}{2}$,

$$E(P, t) = \inf_c \| P(x) - cx^t \|_2^2.$$

For $\gamma \geqslant -\frac{1}{2}$,

$$E_\gamma(P) = \inf\{E(P, t) : t \geqslant \gamma,\ t > -\tfrac{1}{2}\}$$

$$M_\gamma(P) = \sup\{t :\ E(P, t) = E_\gamma(P),\ t \geqslant \gamma,\ t > -\tfrac{1}{2}\}$$

$$M_{n,\gamma} = \sup\{M_\gamma(P) : P \not\equiv 0,\ P \in \pi_n\}$$

$$M_n = M_{n,n}.$$

It is easy to show (Lemma (4) below) that $M_n < \infty$. Therefore M_n can be defined directly by the following two properties:

(i) If P is a polynomial of degree $\leqslant n$, and if among all quasi-monomial cx^s, c real, s real, $s \geqslant n$, the quasi-monomial $c_0 x^{s_0}$ provides a best approximation to P in L_2 [0, 1], then $s_0 \leqslant M_n$.

(ii) If $K < M_n$, there exists a polynomial P of degree $\leqslant n$ and a quasi-monomial of degree greater than K which is a best approximation to P in L_2 [0, 1] among all quasi-monomials of degree $\geqslant n$.

Our main problem is to give an estimate for M_n.

THE MAIN THEOREM. *For all $n > 1$ we have,*

$$\tfrac{1}{4}(n+1)^3 \leqslant M_n \leqslant 6(n+1)^3.$$

Remarks. (1) $M_\gamma(P)$ is well defined, since the set $\{t : t \in R, t \geqslant \gamma; E(P;t) = E_\gamma(P)\}$ is non-empty. This follows from Lemma (4) below because $E(P;t)$ attains its infimum $E_\gamma(P)$.

It also follows from Lemma (4) that the supremum in the definition of $M_\gamma(P)$ is attained.

(2) $M_{n,\gamma}$ is finite. This follows from Lemma (4), and our proof of the main theorem depends essentially on this fact.

(3) Since $M_{n,\gamma}$ is an increasing function of γ, the inequality

$$M_{n,\gamma} \leqslant 6(n+1)^3$$

holds for all γ, $-\frac{1}{2} \leqslant \gamma < n$, in particular for $\gamma = 0$ and $\gamma = -\frac{1}{2}$. However, it is an open problem whether $M_{n,0}$ or $M_{n,-1/2}$ are still bounded below by a constant multiple of $(n+1)^3$.

(4) For fixed n and γ, $M_{n,\gamma}$ can be computed numerically. For example, let

$$V_n(x) = \frac{\sqrt{x}}{\prod_{k=0}^n [(2k+1)x+1]},$$

let T_n be the unique monic polynomial satisfying

$$\| V_n T_n \|_{[0,1]} = \inf\left\{ \left\| V_n(x)\left(x^n - \sum_{i=0}^{n-1} c_i x^i\right) \right\|_{[0,1]} : (c_0, ..., c_{n-1}) \in R^n \right\},$$

and let $\mu_n = \min\{\xi \in [0,1] : |V_n(\xi)\, T_n(\xi)| = \| V_n T_n \|_{[0,1]}\}$. Then it is easy to see, using a lemma of Saff and Varga [8] (stated before Lemma (6) below), that

$$\mu_n = \mu_{n,0},$$

where $\mu_{n,0}$ is as defined in Corollary (2); namely

$$\mu_{n,0}=\inf\{\xi\in[0,1]: |P(\xi)\,V_n(\xi)|=\|PV_n\|_{[0,1]}$$

for some $P\in(\pi_n-\{0\})\}$. Therefore, using relation (17) from Corollary (2), we obtain

$$M_{n,0}=\frac{1}{2}\left(\frac{1}{\mu_n}-1\right).$$

Using a Remes algorithm, we can determine the polynomial T_n and thus μ_n and $M_{n,0}$. This way we obtain the following numerical values:

n	$M_{n,0}$	$M_{n,0}/(n+1)^3$
1	5.82	0.7276
2	21.81	0.8076
3	52.63	0.8223
4	103.09	0.8167
5	175.15	0.8109
6	276.04	0.8048
7	409.06	0.7989
8	578.45	0.7935
9	788.51	0.7885
10	1043.80	0.7842
11	1348.48	0.7804
12	1706.95	0.7769
13	2123.54	0.7739
14	2602.57	0.7711
15	3148.37	0.7686
16	3765.28	0.7664

III. Proof of the Main Theorem

1. *General Remarks*

There are two crucial steps in the proof of the main theorem.

The first crucial step (Theorem (1)) is the transcription of the original problem as stated above into the following form:

Let

$$W_n(x)=\frac{\sqrt{2x+1}}{\prod_{k=0}^{n}(x+k+1)},$$

$$\xi_n=\sup\{\xi(P,W_n): P\not\equiv 0,\ P\in\pi_n\},$$

where

$$\xi(P, W_n) = \max\{\xi \geqslant n : |W_n(\xi)\, P(\xi)| = \|W_n P\|_{[n,\infty)}\}.$$

Give an estimate for ξ_n.

This step makes possible the use of the techniques developed in the studies on incomplete polynomials. For example, with some modification and adaptation (including a correction) it is possible to follow the method of Lorentz [5], i.e., by using:

(i) the formula

$$\lim_{r \to 1^-} \frac{\mathscr{P}_r(f; t) - f(t)}{1-r} = -\tilde{f}'(t),$$

valid for smooth periodic function f (here $\mathscr{P}_r(f; t)$ is the Poisson transform of f, and $\tilde{f}$ is the conjugate function of f); and

(ii) the following lemma of Rahman and Schmeisser [6].

LEMMA. *Let $P \in \pi_n$ and let $M(x)$ be a continuous positive function on some interval $[a, b]$ such that*

$$|P(x)| \leqslant M(x) \qquad \textit{for all} \quad x \in [a, b].$$

Then, for $c > b$, we have

$$|P(c)| \leqslant \frac{1}{r^n} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2) \log M((b+a)/2 - ((b-a)/2) \cos t)}{1 + 2r \cos t + r^2}\, dt\right\},$$

where $r = \delta - \sqrt{\delta^2 - 1}$, $\delta = (2c - b - a)/(b - a)$.

The second crucial step resides in Lemma (7), which we have derived by explicitly finding the Chebyshev polynomials for the weight x on $[0, 1]$ (Lemma (6)).

Lemma (7) makes it possible to replace the weight x by the weight $W_n(x)$ and so to construct a counterexample, which gives a lower bound for M_n.

One of the main results proved by Lorentz in [5] is the following:

THEOREM. *For each $0 < \theta < 1$, there is $0 < \delta < 1$ with the following property. If polynomials*

$$P_n(x) = \sum_{k=s}^{n} a_k x^k, \qquad s \geqslant n\theta, \tag{1}$$

defined for infinitely many n, satisfy $|P_n(x)| \leqslant M$, $0 \leqslant x \leqslant 1$, *then* $P_n(x) \to 0$ *uniformly on* $[0, \delta]$.

The set of all polynomials of form (1) will be denoted by I_θ (this notation was used by Saff in [7].)

Lorentz defined $\Delta(\theta)$ to be the supremum of numbers δ, for which the above theorem is true, and he proved that $\Delta(\theta) \geqslant \theta^2$ [5].

Our proof of the upper bound in the main theorem is an adaptation of Lorentz's proof of the inequality $\Delta(\theta) \geqslant \theta^2$ in [5]. However, that proof, as presented in [5] contains, in its final part, a serious gap (or error). Namely, Lorentz shows that an estimate of the type

$$\lim_{n \to \infty} \frac{1}{n} \log |P_n(x)| \leqslant A(r, a) + o(1-r), \qquad r \to 1, \tag{$*$}$$

holds for a sequence of polynomials P_n, for each $0 < a < \theta^2$, where $0 < r < 1$, $A(r, a) < 0$, and $r \to 1$ as $x \to a$. He concludes, "it follows that for each $a < \theta^2$ and some $\varepsilon > 0$, $P_n(x) \to 0$ uniformly on $[a - \varepsilon, a]$. By "induction in the continuum" we obtain $P_n(x) \to 0$ on $[0, \theta^2)$." (There is a mistake in this which is easy to correct. Polynomials P_n converge uniformly on the interval $[a - \varepsilon, a - \varepsilon/2]$, but not necessarily on $[a - \varepsilon, a]$.)

The serious gap (or the error) is in the implicit claim that ε can be chosen independently of a. Analyzing the derivation of formula ($*$), we see that the $o(1-r)$ term in ($*$) comes from estimates of derivatives of functions $\log((1+a)/2 + (1-a/2) \cos t)$, and so it is not even plausible that the o-term is uniform for a in a neighborhood of zero.

We can, however, salvage this proof of Lorentz in the following way: first we apply another theorem of Lorentz (Theorem 5 in the same article, [5], which we stated above) to show there exists $\delta > 0$ such that $P_n(x)$ converges uniformly to zero on $[0, \delta]$; then we show that for all a, $\delta \leqslant a < \theta^2$, there is $\varepsilon > 0$ (independent of a and dependent only on δ) such that P_n converges uniformly to zero on each interval $[a - \varepsilon, a - \varepsilon/2]$.

This method in which we corrected Lorentz's proof was essential for our proof of the upper bound in the main theorem.

2. *Preliminary Results for the Proof of the Upper Bound.*

One step of Lorentz's proof of the inequality $\Delta(\theta) \geqslant \theta^2$ is showing that if

$$f(t) = \log\left(\frac{1-a}{2} \cos t + \frac{1+a}{2}\right), \qquad a > 0, \tag{2}$$

then

$$\mathscr{P}_r(f; t) = f(t) - (1-r)(\tilde{f})'(t) + o(1-r), \qquad r \to 1^- \tag{3}$$

and

$$(\tilde{f})'(x)=\frac{1}{2\pi}\int_0^{\pi}[f'(x-t)-f'(x+t)]\cot\frac{t}{2}\,dt, \tag{4}$$

where $\mathscr{P}_r(f;t)$ is the Poisson transform of f at t,

$$\mathscr{P}_r(f;t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)f(\theta)}{1-2r\cos(\theta-t)+r^2}\,d\theta,$$

and $\tilde{f}$ is the conjugate function of f,

$$\tilde{f}(x)=\frac{1}{2\pi}\int_0^{\pi}[f(x-t)-f(x+t)]\cot\frac{t}{2}\,dt.$$

Lorentz's proof of (3) and (4) works, not only for the particular function (2), but for a wide class of functions. However, for the proof of the main theorem we need more precise results, including an estimate of the remainder term in the following lemma.

LEMMA (1). *If f is a periodic function of period 2π and has a bounded fourth derivative, then for all t and all $r\in[0,1)$ we have*

$$\mathscr{P}_r(f;t)=f(t)-(1-r)(\tilde{f})'(t)+(1-r)^2H(f;r,t), \tag{5}$$

where $|H(f;r,t)|\leqslant M=\max_t|f^{(4)}(t)|$.

Proof. Let $c_k=(1/2\pi)\int_0^{2\pi}f(t)\,e^{-ikt}\,dt$, $k=0,\pm1,\pm2,\ldots$. Integrating by parts four times we obtain,

$$|c_k|\leqslant\frac{1}{2\pi\,|k|^4}\int_0^{2\pi}|f^{(4)}(t)|\,dt\leqslant\frac{M}{|k|^4},\qquad k=\pm1,\pm2,\ldots.$$

So if we let $C_k(t)$ be the general term of the Fourier series of f ($C_k(t)=c_0$ if $k=0$, $C_k(t)=c_ke^{ikt}+c_{-k}e^{-ikt}$, $k>0$), then

$$|C_k(t)|\leqslant\frac{2M}{k^4},\qquad k=1,2,\ldots$$

and

$$f(t)=\sum_{k=0}^{\infty}C_k(t).$$

Since $\mathscr{P}_r(f; t) = \sum_{k=0}^{\infty} r^k C_k(t)$ for $r \in [0, 1)$, then for $r \in [0, 1)$ we have

$$\frac{\mathscr{P}_r(f; t) - f(t)}{1-r} = -\sum_{k=1}^{\infty} C_k(t) \frac{r^k - 1}{r-1}$$

$$= -\sum_{k=1}^{\infty} (r^{k-1} + \cdots + r + 1)\, C_k(t). \tag{6}$$

Since f is differentiable, $\tilde{f}$ is bounded, and so it is integrable. Therefore, (see [11], p. 156), the Fourier series of $\tilde{f}$ is

$$-i \sum_{-\infty}^{\infty} (\operatorname{sgn} k)\, c_k e^{ikt},$$

and since $|c_k| \leqslant M/|k|^4$, this Fourier series converges uniformly, and so we have

$$\tilde{f}(t) = -i \sum_{-\infty}^{\infty} (\operatorname{sgn} k)\, c_k e^{ikt} \qquad \text{at every } t.$$

Also, the differentiated series is uniformly cnvergent, thus

$$(\tilde{f})'(t) = \sum_{-\infty}^{\infty} |k|\, c_k e^{ikt} = \sum_{k=1}^{\infty} k C_k(t). \tag{7}$$

From (6) and (7), and for $r \in [0, 1)$, we have

$$\frac{\mathscr{P}_r(f; t) - f(t)}{1-r} + (\tilde{f})'(t) = -\sum_{k=2}^{\infty} [r^{k-1} + \cdots + r - (k-1)]\, C_k(t).$$

We write

$$\begin{aligned} &r^{k-1} + \cdots + r - (k-1) \\ &\quad = (r^{k-1} - 1) + \cdots + (r-1) \\ &\quad = (r-1)\{r^{k-2} + 2r^{k-3} + \cdots + (k-2)\, r + k - 1\} \\ &\quad = (r-1)\, a_k(r). \end{aligned}$$

Therefore, for $r \in [0, 1)$ we have

$$\frac{\mathscr{P}_r(f; t) - f(t)}{1-r} + (\tilde{f})'(t) = (1-r) \sum_{k=2}^{\infty} a_k(r)\, C_k(t). \tag{8}$$

Since $|a_k(r)\, C_k(t)| \leqslant ((k-1)/k^3)\, M$ for $r \in [0, 1)$, the series in (8) converges uniformly in t and r.

Let $H(f; r, t) = \sum_{k=2}^{\infty} a_k(r)\, C_k(t)$.

LEMMA (2). *If f is periodic of period 2π and has a bounded second derivative, then* (4) *holds, i.e.,*

$$(\tilde{f})'(x) = \frac{1}{2\pi} \int_0^{\pi} [f'(x-t) - f'(x+t)] \cot \frac{t}{2}\, dt.$$

Proof. From the definition of $\tilde{f}$, we have

$$\begin{aligned}\tilde{f}(x) &= \frac{1}{2\pi} \int_0^{\pi} [f(x-t) - f(x+t)] \cot \frac{t}{2}\, dt \\ &= -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f(x+t) \cot \frac{t}{2}\, dt.\end{aligned}$$

So integrating by parts and noting that $[f(x+\varepsilon) - f(x-\varepsilon)] \log \sin(\varepsilon/2) \to 0$, as $\varepsilon \to 0$ we get

$$\tilde{f}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x+t) \log \left| \sin \frac{t}{2} \right| dt.$$

It follows that

$$\begin{aligned}&\frac{\tilde{f}(x+h) - \tilde{f}(x)}{h} \\ &\qquad = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f'(x+h+t) - f'(x+t)}{h} \log \left| \sin \frac{t}{2} \right| dt.\end{aligned}$$

Since

$$\left| \frac{f'(x+h+t) - f'(x+t)}{h} \right| \leqslant \max |f''(t)| = M,$$

and $\log |\sin(t/2)|$ is integrable, then by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\lim_{h \to 0} \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h} \\ &\qquad = \frac{1}{\pi} \int_{-\pi}^{\pi} \lim_{h \to 0} \frac{f'(x+t+h) - f'(x+t)}{h} \log \left| \sin \frac{t}{2} \right| dt \\ &\qquad = \frac{1}{\pi} \int_{-\pi}^{\pi} f''(x+t) \log \left| \sin \frac{t}{2} \right| dt\end{aligned}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f''(x+t) \log \left| \sin \frac{t}{2} \right| dt$$

$$= \lim_{\varepsilon \to 0} -\frac{1}{2\pi} \left(\int_{-\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f'(x+t) \cot \frac{t}{2}\, dt$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x+t) \cot \frac{t}{2}\, dt$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} [f'(x-t) - f'(x+t)] \cot \frac{t}{2}\, dt.$$

Remark. Lemma (2) is just saying that, under certain conditions, the derivative of the conjugate function is the conjugate of the derivative of the function, i.e.,

$$\tilde{f}'(t) = (\tilde{f})'(t) \qquad \text{at every } t.$$

LEMMA (3). *Let $f(t) = \log(A - B\cos t)$, $A > |B|$. Then for $r \in [0, 1)$ we have*

$$\mathscr{P}_r(f; \pi) = \log(A+B) + (1-r)\left(\sqrt{\frac{A-B}{A+B}} - 1\right)$$
$$+ (1-r)^2 H(A, B; r), \tag{9}$$

where $|H(A, B; r)| \leqslant C\delta^{-4}$ (C is an absolute constant) provided $1 - |B|/A \geqslant \delta$, $\delta > 0$.

Proof. Since f satisfies the hypothesis of Lemma (1), then by (5) we have

$$\mathscr{P}_r(f; \pi) = f(\pi) - (1-r)(\tilde{f})'(\pi) + (1-r)^2 H(f; r, \pi),$$

and

$$|H(f; r, \pi)| \leqslant \max_t |f^{(4)}(t)|.$$

We will use $H(A, B; r)$ in place of $H(f; r, \pi)$ since $H(f; r, \pi)$ depends on A, B and r.

Since $f(\pi) = \log(A+B)$, (9) will follow if we show that $(\tilde{f})'(\pi) = 1 - \sqrt{(A-B)/(A+B)}$, and

$$\max_t |f^{(4)}(t)| \leqslant C\delta^{-4} \qquad \text{whenever } 1 - \frac{|B|}{A} \geqslant \delta.$$

By Lemma (2),

$$
\begin{aligned}
(\tilde{f})'(\pi) &= \frac{1}{2\pi}\int_0^{\pi} [f'(\pi-t)-f'(\pi+t)] \cot\frac{t}{2}\,dt \\
&= \frac{B}{\pi}\int_0^{\pi} \frac{\sin t}{A+B\cos t}\cot\frac{t}{2}\,dt \\
&= \frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{1+u^2}-\frac{1}{(A+B)/(A-B)+u^2}\right)du \\
&= 1-\sqrt{\frac{A-B}{A+B}}.
\end{aligned}
$$

Since

$$
f'(t)=\frac{\lambda\sin t}{1-\lambda\cos t}, \qquad \lambda=\frac{B}{A},
$$

we get

$$
f^{(4)}(t)=\frac{P(\lambda)}{(1-\lambda\cos t)^4},
$$

where $P(\lambda)$ is a polynomial in λ (with coefficients trigonometric polynomials in t). Then since $|\lambda|<1$, we have $\max|f^{(4)}(t)|\leqslant C(1-|\lambda|)^{-4}$.

So if $1-|B|/A\geqslant\delta$, then $|H(A,B;r)|\leqslant C\delta^{-4}$.

THEOREM (1). *Let $P\in\pi_n$. There exists a polynomial $Q\in\pi_n$ such that, for $t>-\frac{1}{2}$*

$$
E(P;t)=\|P\|_2^2-\{U_n(t)\,Q(t)\}^2, \tag{10}
$$

where

$$
U_n(t)=\frac{\sqrt{2t+1}}{\prod_{k=0}^{n}(t+k+1)}.
$$

Moreover,

(i) *the mapping $P\to Q$ is a bijection on π_n.*

(ii) $Q(x)=\sum_{k=0}^{n}(P^{(k)}(0)/k!)\prod_{i=0,i\neq k}^{n}(x+i+1)$.

(iii) $P(x)=\sum_{k=0}^{n}(-1)^k(Q(-k-1)/k!(n-k)!)\,x^k$.

COROLLARY (1). *There are bijections $P \to R$ and $P \to S$ on π_n such that*

$$E\left(P; \frac{x-1}{2}\right) = \| P \|_2^2 - 4^{n+1} \{W_n(x)\, R(x)\}^2, \qquad x > 0, \tag{11}$$

where

$$W_n(x) = \frac{\sqrt{x}}{\prod_{k=0}^{n} (x+2k+1)} \qquad \textit{and} \qquad R(x) = Q\left(\frac{x-1}{2}\right).$$

$$E\left(P, \frac{1/y-1}{2}\right) = \| P \|_2^2 - 4^{n+1} \{V_n(y)\, S(y)\}^2, \qquad y > 0, \tag{12}$$

where

$$V_n(y) = \frac{\sqrt{y}}{\prod_{k=0}^{n} [(2k+1)\, y + 1]} \qquad \textit{and} \qquad S(y) = y^n R\left(\frac{1}{y}\right).$$

Proof of Theorem (1). Let $P(x) = \sum_{k=0}^{n} a_k x^k$ and $t > -\frac{1}{2}$. By definition, $E(P; t)$ is the square of the the L_2-distance from P to the subspace spanned by x^t, so by the well-known distance formula in inner product spaces, we have

$$E(P; t) = \frac{G(x^t, P)}{G(x^t)},$$

where $G(f_1, ..., f_m)$ is the Gram determinant on $\{f_1, ..., f_m\}$. This gives

$$\begin{aligned} E(P; t) &= \frac{\| P \|_2^2 \, \| x^t \|_2^2 - \langle x^t, P \rangle^2}{\| x^t \|_2^2} \\ &= \| P \|_2^2 - (2t+1) \left(\sum_{k=0}^{n} \frac{a_k}{t+k+1} \right)^2. \end{aligned} \tag{13}$$

We write

$$\sum_{k=0}^{n} \frac{a_k}{t+k+1} = \frac{1}{\prod_{k=0}^{n} (t+k+1)} \sum_{k=0}^{n} a_k \prod_{\substack{i=0 \\ i \neq k}}^{n} (t+i+1). \tag{14}$$

Then from (13) and (14), since $a_k = P^{(k)}(0)/k!$, the formulas in (10) and (ii) follow.

In (ii), if we let $x = -j-1$, we get

$$Q(-j-1) = \frac{P^{(j)}(0)}{j!} \prod_{\substack{i=0 \\ i \neq j}}^{n} (i-j) = (-1)^j \, P^{(j)}(0)(n-j)!,$$

from which (iii) follows.

Finally, the bijection follows from (ii) and (iii).

LEMMA (4). *Let* $E_S(P) = \inf\{E(P; t) : t \in S\}$.

(i) *For any set* S, $S \subseteq \{t : t > -\frac{1}{2}\}$, *there are constants* $c_n = c_n(S)$ *and* $L_n = L_n(S)$ *such that if* $P \in \pi_n$ *and* $P \not\equiv 0$, *then*

$$E(P; t) > E_S(P), \qquad \text{for} \quad t \in S \cap [\{x : x < c_n\} \cup \{x : x > L_n\}].$$

(ii) *If* S *is relatively closed with respect to* $\{t : t > -\frac{1}{2}\}$, *then for every* $P \in \pi_n$ *and* $P \not\equiv 0$, *there is* $t \in S$ *such that*

$$E(P; t) = E_S(P);$$

i.e., the best approximation exists.

Proof. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ and $P \not\equiv 0$. We may assume that $\| P \|_2 = 1$, so there is a constant $K(n)$ such that

$$| a_k | \leqslant K(n) \qquad \text{for } k = 0, ..., n.$$

If

$$F_P(t) = \sqrt{2t+1} \left| \sum_{i=0}^{n} \frac{a_i}{t+i+1} \right|,$$

then

$$| F_P(t) | \leqslant (n+1)\, K(n) \frac{\sqrt{2t+1}}{t+1} \qquad \text{for} \quad t > -\frac{1}{2}.$$

The last inequality implies that $\lim_{t \to -1/2} F_P(t) = 0$ uniformly for $P \in \pi_n$ and $\| P \|_2 = 1$, and $\lim_{t \to \infty} F_P(t) = 0$ uniformly for $P \in \pi_n$ and $\| P \|_2 = 1$. Therefore, there exist $c_n = c_n(S)$ and $L_n = L_n(S)$ such that

$$| F_P(t) | < \| F_P \|_S \qquad \text{if} \quad t \in S \cap [\{x : x < c_n\} \cup \{x : x > L_n\}] \tag{15}$$

(recall that $\| F_P \|_S = \sup_{t \in S} | F_P(t) |$).

Since, $E_S(P) = \inf\{E(P; t) : t \in S\}$, then by (13) we have,

$$E_S(P) = \| P \|_2^2 - \| F_P \|_S^2 = 1 - \| F_P \|_S^2 \leqslant 1 - (F_P(t))^2 = E(P, t). \tag{16}$$

Thus, (i) follows from (15) and (16).

If S is relatively closed with respect to $\{t : t > -\frac{1}{2}\}$, then since $F_P \not\equiv 0$ and $\lim_{t \to -1/2} F_P(t) = \lim_{t \to \infty} F_P(t) = 0$, (ii) follows by the continuity of F_P.

COROLLARY (2). *For $\gamma > -\frac{1}{2}$ and $P \in \pi_n$, let*

$$\lambda_\gamma(P) = \max\{\xi : |U_n(\xi)\, P(\xi)| = \| U_n P \|_{[\gamma,\infty)},\ \xi \geq \gamma\},$$

$$\xi_\gamma(P) = \max\{\xi : |W_n(\xi)\, P(\xi)| = \| W_n P \|_{[2\gamma+1,\infty)},\ \xi \geq 2\gamma + 1\},$$

$$\mu_\gamma(P) = \min\left\{\xi : V_n(\xi)\, P(\xi)| = \| V_n P \|_{[0,\,1/(2\gamma+1)]},\ 0 \leq \xi \leq \frac{1}{2\gamma+1}\right\},$$

$$\lambda_{n,\gamma} = \sup\{\lambda_\gamma(P) : P \not\equiv 0,\ P \in \pi_n\},$$

$$\xi_{n,\gamma} = \sup\{\xi_\gamma(P) : P \not\equiv 0,\ P \in \pi_n\},$$

and

$$\mu_{n,\gamma} = \inf\{\mu_\gamma(P) : P \not\equiv 0,\ P \in \pi_n\},$$

where U_n, W_n, and V_n are as defined in Theorem (1) *and Corollary* (1). *Then*

$$M_{n,\gamma} = \lambda_{n,\gamma} = \frac{\xi_{n,\gamma} - 1}{2} = \frac{1}{2}\left(\frac{1}{\mu_{n,\gamma}} - 1\right). \tag{17}$$

Remark. By the lemma of Saff and Varga [8] (stated before Lemma (6) below), we can replace the sup in the definition of $\lambda_{n,\alpha}$ and $\xi_{n,\alpha}$ by max and the inf in the definition of $\mu_{n,\alpha}$ by min.

Proof. We show the first equality in (17), the rest is obvious.
By definition $E_\gamma(P) = \max\{E(P, t) : t \geq \gamma,\ t > -\frac{1}{2}\}$.
So by (10), we have

$$E_\gamma(P) = E(P; t) \qquad \text{if and only if } |U_n(t)\, Q(t)| = \| U_n Q \|_{[\gamma,\infty)}.$$

Since $M_{n,\gamma} = \sup\{\xi \geq \gamma : E(P, \xi) = E_\gamma(P)$ for some $P \in (\pi_n - \{0\})\}$, and since the mapping $P \to Q$ is a bijection on π_n by Theorem (1), the first equality in (17) follows.

LEMMA (5). *Let $a = 2n + 1$ and $b \geq 12(n+1)^3$.*

(i) *There exists $\mu_n > 0$ such that,*

$$\frac{1}{2} - \sum_{k=0}^{n} \sqrt{\frac{a + 2k + 1}{b + 2k + 1}} \geq \mu_n. \tag{18}$$

(ii) *For $c > b$, $\delta = (2c - b - a)/(b - a)$, and $r = \delta - \sqrt{\delta^2 - 1}$, we have*

$$1 - r \leq \sqrt{c - b}. \tag{19}$$

Proof. (i) Let $b_n = 12(n+1)^3$, and

$$\mu_n = \frac{1}{2} - \sum_{k=0}^{n} \sqrt{\frac{a+2k+1}{b_n+2k+1}}.$$

We have

$$\begin{aligned}\sum_{k=0}^{n} \sqrt{\frac{a+2k+1}{b_n+2k+1}} &\leqslant \sqrt{\frac{2}{b_n}} \sum_{k=0}^{n} \sqrt{n+k+1} \\ &\leqslant \sqrt{\frac{2}{b_n}} \int_0^{n+1} \sqrt{x+n+1}\, dx = \frac{8-\sqrt{8}}{3\sqrt{b_n}} \sqrt{(n+1)^3} \\ &= \frac{8-\sqrt{8}}{6\sqrt{3}} \leqslant 0.498\end{aligned}$$

so $\mu_n \geqslant 0.002$. Since

$$\sum_{k=0}^{n} \sqrt{\frac{a+2k+1}{b+2k+1}}$$

is decreasing in b, (18) follows.

(ii) $1-r = 1-\delta+\sqrt{\delta^2-1} = \sqrt{\delta-1}\,(\sqrt{\delta+1}-\sqrt{\delta-1}) = \sqrt{\delta-1}\,(2/(\sqrt{\delta+1}+\sqrt{\delta-1}))$. Since $\delta-1 = 2(c-b)/(b-a)$ and $\delta+1 = (c-a)/(b-a)$, we have

$$1-r = \frac{2}{\sqrt{c-a}+\sqrt{c-b}} \sqrt{c-b}.$$

But $c-a \geqslant 12(n+1)^3 - 2n - 1 > 4$, so $1-r \leqslant \sqrt{c-b}$.

3. *Proof of the Upper Bound*

We will prove that

$$M_n \leqslant 6(n+1)^3. \tag{20}$$

Recall that $M_n = M_{n,n}$.

By Corollary (2), $M_{n,n} = (\xi_{n,n}-1)/2$. So we need to show that

$$\xi_{n,n} \leqslant 12(n+1)^3 + 1,$$

where $\xi_{n,n}$ is as defined in Corollary (2); namely, $\xi_{n,n} = \max\{\xi : |W_n(\xi)| = \|W_n P\|_{[2n+1,\infty)}$ for some $P \in (\pi_n - \{0\})\}$, where

$$W_n(x) = \frac{\sqrt{x}}{\prod_{k=0}^{n}(x+2k+1)}.$$

Thus, it is clear that, to prove (20), it is enough to prove the following:

If $P \in \pi_n$, $F_n(x) = W_n(x)\, P(x)$, and $\| F_n \|_{[2n+1,\infty)} = 1$ then $|F_n(\xi)| < 1$ for $\xi > 12(n+1)^3$. $\quad (**)$

The proof of $(**)$ will follow from (i) and (ii) below.

(i) By Lemma (4), there is L_n such that if $\xi > L_n$, then $|F_n(\xi)| < 1$. Since if $L_n \leqslant 12(n+1)^3$, (20) holds, so we assume that $L_n > 12(n+1)^3$.

(ii) There is $\varepsilon_n > 0$ dependent only on n such that if $12(n+1)^3 \leqslant b \leqslant L_n$ and $c \in (b, b+\varepsilon_n)$, then $|F_n(c)| < 1$. (Observe that

$\{c\colon 12(n+1)^3 < c \leqslant L_n\} \subseteq U\{\{c\colon b < c < b+\varepsilon_n\}\colon 12(n+1)^3 \leqslant b \leqslant L_n\}$.)

So we let F_n be as in $(**)$ and L_n as in (i), then we have

$$|F_n(x)| \leqslant 1, \qquad x \geqslant 2n+1$$

i.e.,

$$|P(x)| \leqslant M(x), \qquad x \geqslant 2n+1$$

where

$$M(x) = \frac{1}{W_n(x)} = x^{-1/2} \prod_{k=0}^{n} (x+2k+1).$$

In particular for b, $12(n+1)^3 \leqslant b \leqslant L_n$,

$$|P(x)| \leqslant M(x), \qquad x \in [2n+1, b].$$

By the Rahman–Schmeisser lemma (stated above), we have for $c > b$,

$$|P(c)| \leqslant \frac{1}{r^n} \exp\left\{\mathscr{P}_r\!\left(\log M\left(\frac{b+a}{2} - \frac{b-a}{2}\cos(\cdot)\right); \pi\right)\right\}, \tag{21}$$

where $r = \delta - \sqrt{\delta^2 - 1}$, $\delta = (2c-a-b)/(b-a)$, and $a = 2n+1$. Since

$$\begin{aligned} &\log M\left(\frac{b+a}{2} - \frac{b-a}{2}\cos t\right) \\ &\quad = -\frac{1}{2}\log\left(\frac{b+a}{2} - \frac{b-a}{2}\cos t\right) \\ &\qquad + \sum_{k=0}^{n} \log\left(\frac{b+a}{2} + 2k+1 - \frac{b-a}{2}\cos t\right), \end{aligned}$$

then if we set,

$$f_{0,b}(t) = -\frac{1}{2}\log\left(\frac{b+a}{2} - \frac{b-a}{2}\cos t\right) \qquad \text{and}$$

$$f_{j+1,b}(t) = \log\left(\frac{b+a}{2} + 2j + 1 - \frac{b-a}{2}\cos t\right) \qquad \text{for} \quad j = 0, ..., n,$$

we have

$$\log M\left(\frac{b+a}{2} - \frac{b-a}{2}\cos t\right) = \sum_{k=0}^{n+1} f_{k,b}(t),$$

so we have

$$\mathscr{P}_r\left(\log M\left(\frac{b+a}{2} - \frac{b-a}{2}\cos(\cdot)\right); \pi\right) = \sum_{k=0}^{n+1} \mathscr{P}_r(f_{k,b}; \pi). \tag{22}$$

Each $f_{k,b}$ is of the form

$$f_{k,b}(t) = C_k \log(A_k - B_k \cos t),$$

for some A_k, B_k, and C_k, where A_k and B_k depend on b. In particular,

$$\begin{aligned} &C_0 = -\frac{1}{2}, \qquad A_0 = \frac{b+a}{2}, \qquad B_0 = \frac{b-a}{2} \\ &C_k = 1, \qquad A_k = \frac{b+a}{2} + 2k - 1, \qquad B_k = \frac{b-a}{2} \qquad \text{if} \quad k = 1, ..., n+1 \end{aligned} \tag{23}$$

since $B_k = (b-a)/2$ and $A_k \geqslant (b+a)/2$ for $k = 0, 1, ..., n+1$; then

$$1 - \frac{B_k}{A_k} \geqslant 1 - \frac{b-a}{b+a} = \frac{2a}{b+a} \qquad \text{for} \quad k = 0, ..., n+1. \tag{24}$$

But $b \leqslant L_n$, so if we let $\delta_n = 2a/(L_n + a)$, then

$$1 - \frac{B_k}{A_k} \geqslant \delta_n > 0.$$

Thus, by (9) we have

$$\mathscr{P}_r(f_{k,b}; \pi) = C_k\left[\log(A_k + B_k) + (1-r)\left(\sqrt{\frac{A_k - B_k}{A_k + B_k}} - 1\right)\right.$$
$$\left. + (1-r)^2 H(A_k, B_k; r)\right], \tag{25}$$

where $|H(A_k, B_k; \pi)| \leqslant C\delta_n^{-4}$.

So if we let $K_n = (n+2)\, C\delta_n^{-4}$, then substituting (25) in (22) gives (notice that K_n is a constant that depends only on n),

$$\begin{aligned}
&\mathscr{P}_r\left(\log M\left(\frac{b+a}{2}-\frac{b-a}{2}\cos(\cdot)\right);\pi\right)\\
&\quad\leqslant \sum_{k=0}^{n+1} C_k\left[\log\left(A_k+B_k\right)+(1-r)\left(\sqrt{\frac{A_k-B_k}{A_k+B_k}}-1\right)\right]\\
&\quad\quad+(1-r)^2 K_n.
\end{aligned}$$

Substituting the values of A_k, B_k, and C_k from (23) in the last inequality gives

$$\begin{aligned}
&\mathscr{P}_r\left(\log M\left(\frac{b+a}{2}-\frac{b-a}{2}\cos(\cdot)\right);\pi\right)\\
&\quad\leqslant -\log\sqrt{b}+\sum_{k=0}^{n}\log(b+2k+1)+(1-r)^2 K_n\\
&\quad\quad+(1-r)\left[-\frac{1}{2}\sqrt{\frac{a}{b}}-n-\frac{1}{2}+\sum_{k=0}^{n}\sqrt{\frac{a+2k+1}{b+2k+1}}\right].
\end{aligned}\tag{26}$$

Since

$$F_n(c)=P(c)\frac{\sqrt{c}}{\prod_{k=0}^{n}(c+2k+1)},$$

then

$$\log|F_n(c)|\leqslant\log|P(c)|+\log\sqrt{c}-\sum_{k=0}^{n}\log(c+2k+1).$$

Therefore, by (21) and (26), we have for $c > b$,

$$\begin{aligned}
\log|F_n(c)|\leqslant{}& -n\log r+\log\sqrt{\frac{c}{b}}-\sum_{k=0}^{n}\log\left(\frac{c+2k+1}{b+2k+1}\right)\\
&+(1-r)\left[-\sqrt{\frac{a}{b}}-n-\frac{1}{2}+\sum_{k=0}^{n}\sqrt{\frac{a+2k+1}{b+2k+1}}\right]\\
&+(1-r)^2 K_n.
\end{aligned}$$

Using formula (18) in Lemma (5), and removing the negative terms

$-((1-r)/2)\sqrt{a/b}$ and $-\log((c+2k+1)/(b+2k+1))$ for $k=1, 2, ..., n$, then for $c>b$, the last inequality gives

$$\log |F_n(c)| \leqslant -n\log r + \log\sqrt{\frac{c}{b}} - \log\left(\frac{c+1}{b+1}\right)$$
$$+(1-r)(-n-\mu_n)+(1-r)^2 K_n.$$

Since $\sqrt{c/b} \leqslant (c+1)/(b+1)$ for $c \geqslant b \geqslant 1$, the last inequality gives, for $c>b$,

$$\log |F_n(c)| \leqslant -n\log r + (1-r)(-n-\mu_n)+(1-r)^2 K_n. \qquad (27)$$

Now by (19) in Lemma (5), for $c>b\geqslant 12(n+1)^2$, we have $1-r\leqslant\sqrt{c-b}$. So, if $\varepsilon<\frac{1}{4}$ and $c\in(b, b+\varepsilon)$, then $1-r\leqslant\sqrt{\varepsilon}<\frac{1}{2}$, and so $-\log r\leqslant(1-r)+(1-r)^2 K$ where K is a constant independent of ε if $\varepsilon<\frac{1}{4}$.

Using this estimate for $-\log r$ in (27) gives the following: For every $\varepsilon<\frac{1}{4}$ and $b\in[12(n+1)^3, L_n]$, we have for $c\in(b, b+\varepsilon)$,

$$\log |F_n(c)| \leqslant \sqrt{\varepsilon}\,[-\mu_n+\sqrt{\varepsilon}\,(K_n+nK)].$$

From the last inequality, it follows that there is $\varepsilon_n<\frac{1}{4}$ such that if $12(n+1)^3\leqslant b\leqslant L_n$ and $c\in(b, b+\varepsilon_n)$, then $|F_n(c)|<1$.

This completes the proof of the upper bound.

4. *Lemmas for the Proof of the Lower Bound*

In this section we will find a counterexample which proves the lower bound for M_n in the main theorem.

By the following Lemma of Saff and Varga [8] and Corollary (2), the best counterexample would be the Chebyshev polynomial of weight

$$V_n(x)=\frac{\sqrt{x}}{\prod_{k=0}^{n}[(2k+1)x+1]}$$

on the interval $[0, 1/(2n+1)]$.

LEMMA. *Suppose the weight function $W(x)\in C[0, 1]$ satisfies $W(0)=0$ and $W(x)>0$ for $x\in[0, 1]$. For each n, let*

$$P_n^*(x)=P_n^*(W; x)=x^n-\sum_{i=0}^{n-1} c_i^* x^i$$

be the unique extremal polynomial for the Chebyshev problem

$$\inf\left\{\left\| W(x)\left(x^n-\sum_{i=0}^{n-1} c_i x^i\right)\right\|_{[0,1]} : (c_0, ..., c_{n-1})\in R^n\right\},$$

and set

$$\xi_n^* = \min\{x \in (0, 1] : |W(x)\, P_n^*(x)| = \| WP_n^* \|_{[0,1]}\}.$$

If $P(x)$ is any real Lacunary polynomial of the form

$$P(x) = \sum_{i=0}^{n} b_i x^{\mu_i}$$

then

$$|P(x)| \leqslant \frac{\| WP \|_{[0,1]}}{\| WP_n^* \|_{[0,1]}} |P_n^*(x)|, \qquad \text{for all} \quad 0 \leqslant x \leqslant \xi_n^*.$$

Consequently, if $\xi \in (0, 1]$ satisfies $|W(\xi)\, P(\xi)| = \| WP \|_{[0,1]}$, where $P \not\equiv 0$ and is of the above form, then

$$\xi_n^* \leqslant \xi.$$

Unfortunately, it is not easy to find the general formula for the Chebyshev polynomials for the weights V_n, $n \geqslant 1$. For this reason, one can use a simpler weight to work with, which can be replaced by V_n; and this is what we do here, we use the weight $W_n(x) = x$ for all n.

LEMMA (6) (Explicit Form of the Chebyshev Polynomial for the Weight x on $[0, 1]$). *Let $T_n(x) = x^n - \sum_{k=0}^{n-1} c_k^* x^k$ be the unique polynomial such that*

$$\| xT_n(x) \|_{[0,1]} = \inf\left\{ \left\| x\left(x^n - \sum_{k=0}^{n-1} c_k x^k\right)\right\|_{[0,1]} : (c_0, ..., c_{n-1}) \in R^n \right\}.$$

Then

$$T_n(x) = \prod_{i=1}^{n} (x - x_i),$$

where

$$x_i = \frac{1}{2}\left\{1 - \xi_n + (1 + \xi_n) \cos\left(\frac{2i+1}{2(n+1)}\pi\right)\right\}$$

and

$$\xi_n = \frac{1 - \cos((\pi/(2(n+1)))}{1 + \cos((\pi/(2(n+1)))}.$$

Remark. The system $\{x, x^2, ..., x^n\}$ is not a Haar System on the interval

[0, 1]; however, by a generalization of Chebyshev's theorem, the polynomial $xT_n(x)$ in the statement of the lemma is unique and has the alternating property [11, footnote on p. 56].

Proof. Let $e_n = \|xT_n(x)\|_{[0,1]}$. By the alternating property, $|xT_n(x)|$ attains its maximaum e_n at $n+1$ points in (0, 1]. So $xT_n(x)$ has $n+1$ distinct zeros in [0, 1]. Since $xT_n(x)$ has at most $n+1$ zeros, then the $n+1$ distinct zeros of $xT_n(x)$ are contained in [0, 1). It follows that $|xT_n(x)|$ is decreasing on $(-\infty, 0)$, and if c is the largest zero of $xT_n(x)$ then $|xT_n(x)|$ is increasing on (c, ∞). Therefore, $|T_n(1)| = e_n$ and there is a unique point $-\xi_n \in (-\infty, 0)$ such that

$$|\xi_n T_n(-\xi_n)| = e_n. \tag{28}$$

Thus, the polynomials $(\xi_n + x)(1-x)[T_n(x) + xT_n'(x)]^2$ and $e_n^2 - x^2 T_n^2(x)$ have exactly the same zeros.

Since the leading coefficient of $(\xi_n + x)(1-x)[T_n(x) + xT_n'(x)]^2$ is $-(n+1)^2$, then the polynomial $y = xT_n(x)$ satisfies the differential equation

$$(\xi_n + x)(1-x)(y')^2 = (n+1)^2\,(e_n^2 - y^2)), \qquad y(0) = 0. \tag{29}$$

The general solution of (29) is of the form

$$y = \pm e_n \cos\left[(n+1)\arccos\left(\frac{2x + \xi_n - 1}{1 + \xi_n}\right) + c\right],$$

but the right-hand side of the last equation is a polynomial if and only if $c = m\pi$, m is an integer, so

$$y = \pm e_n \cos\left[(n+1)\arccos\left(\frac{2x + \xi_n - 1}{1 + \xi_n}\right)\right]. \tag{30}$$

Since $y(0) = 0$, we have

$$\arccos\left(\frac{\xi_n - 1}{\xi_n + 1}\right) = \frac{2k+1}{2(n+1)}\pi \qquad \text{for some } k \in \{0, 1, ..., n\}.$$

So

$$\xi_n = \frac{1 + \cos(((2k+1)/2(n+1))\,\pi)}{1 - \cos(((2k+1)/2(n+1))\,\pi)} \quad \text{and} \quad \frac{\xi_n - 1}{\xi_n + 1} = \cos\left(\frac{2k+1}{2(n+1)}\pi\right). \tag{31}$$

The $n+1$ zeros of y, $\{x_0, ..., x_n\}$ are given by

$$\frac{2x_i + \xi_n - 1}{\xi_n + 1} = \cos\left(\frac{2i+1}{2(n+1)}\pi\right) \qquad \text{for} \quad i = 0, ..., n.$$

Therefore,

$$x_i = \frac{1}{2}(1+\xi_n)\left\{\cos\left(\frac{2i+1}{2(n+1)}\pi\right) + \frac{1-\xi_n}{1+\xi_n}\right\},$$

and so by (31) we have,

$$x_i = \frac{1}{2}(1+\xi_n)\left\{\cos\left(\frac{2i+1}{2(n+1)}\pi\right) - \cos\left(\frac{2k+1}{2(n+1)}\pi\right)\right\}.$$

Since $x_i > 0$ for $i \neq k$, then for $i \in \{0, ..., n\} \setminus \{k\}$ we have

$$\cos\left(\frac{2i+1}{2(n+1)}\pi\right) > \cos\left(\frac{2k+1}{2(n+1)}\pi\right),$$

so $k = n$, and by (31) we get

$$\xi_n = \frac{1-\cos(\pi/2(n+1))}{1+\cos(\pi/2(n+1)))} \qquad \text{and} \qquad \frac{\xi_n - 1}{\xi_n + 1} = -\cos\left(\frac{\pi}{2(n+1)}\right). \tag{32}$$

LEMMA (7). *If $T_n(x)$ is defined as in Lemma (6) and*

$$t_n = \min\{t \in (0, 1] : |tT_n(t)| = \|xT_n(x)\|_{[0,1]}\},$$

then $t_n \leqslant 3/(n+1)^2$.

Proof. Let $y(x) = xT_n(x)$ and let ξ_n be as in Lemma (6).

By Lemma (6), $x = 0$ is a simple zero of y. So the zeros of the second derivative y'' of y are contained in the interval $[t_n, 1]$, and so y'' is of constant sign on $(-\infty, t_n)$. Since y is a polynomial, $|y|$ is convex on $(-\infty, t_n)$. In particular it is convex on $[-\xi_n, 0]$, so we have

$$\frac{|y(-\xi_n)|}{\xi_n} \geqslant |y'(0)|.$$

Since by (28) we have $|y(-\xi_n)| = e_n$, then the last inequality gives

$$\frac{e_n}{\xi_n} \geqslant |y'(0)|.$$

But by (29) we have $\xi_n(y'(0))^2 = (n+1)^2 e_n^2$, so the last inequality gives

$$\frac{e_n}{\xi_n} \geqslant \frac{(n+1)\, e_n}{\sqrt{\xi_n}}$$

which implies that

$$\xi_n \leqslant \frac{1}{(n+1)^2}. \tag{33}$$

From (30) we obtain that $y'(t) = 0$ if and only if

$$\frac{2t + \xi_n - 1}{1 + \xi_n} = \cos\left(\frac{k\pi}{n+1}\right) \qquad \text{for some} \quad k \in \{0, 1, ..., n\}.$$

It follows, since $t_n = \min\{t : | y'(t)| = 0\}$, that

$$\frac{2t_n + \xi_n - 1}{1 + \xi_n} = -\cos\left(\frac{\pi}{n+1}\right) = -\left(2\cos^2\left(\frac{\pi}{2(n+1)}\right) - 1\right).$$

So by (32) we have

$$\frac{2t_n + \xi_n - 1}{1 + \xi_n} = 1 - 2\left(\frac{1 - \xi_n}{1 + \xi_n}\right)^2,$$

which implies that

$$t_n = \frac{3 - \xi_n}{1 + \xi_n} \xi_n \leqslant 3\xi_n.$$

Finally from (33) we get

$$t_n \leqslant \frac{3}{(n+1)^2}.$$

Remarks. (1) It is easy, of course, from $t_n = ((3 - \xi_n)/(1 + \xi_n))\, \xi_n$ and the expression for ξ_n in Lemma (6) to deduce an exact expression for t_n, from which it follows that

$$t_n \sim \frac{3\pi^2}{16n^2}, \qquad n \to \infty.$$

(2) Let $0 < \theta < 1$, and $n = [1/\theta] + 1$.

Let $P_n(x) = xT_{n-1}(x)$, where $T_{n-1}(x)$ is the Chebyshev polynomial of degree $n - 1$ which is defined in Lemma (6).

It follows from Lemma (7) that

$$\| P_n \|_{[0,1]} = \| P_n \|_{[0, 3\theta^2]}.$$

This may be of interest as a complement to the following result of Saff [7] (since $P_n \in I_\theta$):

THEOREM. *For each* $0 < \theta < 1$,

$$\inf\{\xi(P): P \in I_\theta, P \not\equiv 0\} = \theta^2,$$

where $\xi(P) = \min\{\xi \in [0, 1]: |P(\xi)| = \|P\|_{[0,1]}\}$.

5. *Proof of the Lower Bound*

We will prove that for all $n > 1$,

$$M_n \geqslant \tfrac{1}{4}(n+1)^3.$$

Let $P_n(x) = T_n((2n+1)x)$ and $\lambda_n = t_n/(2n+1)$, where T_n and t_n are as in Lemmas (6) and (7).

Let

$$G_n(x) = \frac{\sqrt{x}\, P_n(x)}{\prod_{k=0}^{n} [(2k+1)x+1]}$$

so

$$G_n(x) = \frac{xP_n(x)}{\sqrt{x} \prod_{k=0}^{n} [(2k+1)x+1]}, \qquad x > 0.$$

Since $\sqrt{x} \prod_{k=0}^{n} [(2k+1)x+1]$ is increasing on $(0, \infty)$, then

$$|G_n(x)| < |G_n(\lambda_n)| \qquad \text{for} \quad x \in \left(\lambda_n, \frac{1}{2n+1}\right]. \tag{34}$$

Therefore,

$$\|G_n\|_{[0, 1/(2n+1)]} = \|G_n\|_{[0, \lambda_n]},$$

and from this it follows that $\mu_n(P_n) \leqslant \lambda_n$, where $\mu_n(P_n)$ is defined as in Corollary (2).

By Lemma (7), $t_n \leqslant 3/(n+1)^2$. So

$$\lambda_n \leqslant \frac{3}{(2n+1)(n+1)^2}. \tag{35}$$

Since $\mu_{n,n} \leqslant \lambda_n$, where $\mu_{n,n}$ is defined as in Corollary (2), then by (35) we get

$$\mu_{n,n} \leqslant \frac{3}{(2n+1)(n+1)^2},$$

and so by (17) in Corollary (2), we have

$$M_{n,n} \geqslant \frac{(2n+1)(n+1)^2}{6} - \frac{1}{2} \geqslant \frac{1}{4}(n+1)^3 \qquad \text{for all} \quad n > 1.$$

This completes the proof since $M_n = M_{n,n}$.

Remark. By Remark (1) following Lemma (7), it is possible to improve the constant $\frac{1}{4}$ in the lower bound for M_n for n large. Namely

$$\liminf_{n \to \infty} \frac{M_n}{n^3} \geqslant \frac{16}{3\pi^2}.$$

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